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The influence of nonlinear material properties and resistance to bending on the development of internal structures

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Abstract—A reexamination of Biot's internal instability analysis, including the effects of bending resistance and nonlinear material properties, shows that internal buckling and oblique localized shearing into bands are the two end-member processes associated with internal instability. The first is a consequence of a high anisotropy which is an intrinsic property of the original unstressed layering or foliation. The second is a consequence of a highly anisotropic material response to perturbations which is induced during the initial uniform compression if material properties are nonlinear. The incremental anisotropy generating the instability is associated with planes of least resistance to shearing in directions parallel to the initial compression in the first case and at 45° to the initial compression in the second case.

INTRODUCTION

THE DEVELOPMENT of internal structures in deformed anisotropic rocks was the subject of a detailed investigation by Cobbold *et al.* (1971). These authors considered the formation of a variety of natural structures including kink bands and sinusoidal folds in rocks as diverse as turbidite sequences, banded gneisses, slates and schists. Their treatment of the mechanics involved in the formation of these structures was founded upon a mathematical analysis proposed by Biot (1965, pp. 192–204) who argued that the deformation behaviour of composite foliated rocks can be analysed and discussed in terms of their average rheological properties. It is through the various applications of Biot's analysis (Johnson 1970, Cobbold *et al.* 1971, Sowers 1973, Cosgrove 1976) that structural geologists have obtained a mechanical understanding of many structures common to foliated rocks.

Some investigations concerned more specifically with the folding of multilayers have developed upon quite different lines; by considering each layer as a discrete unit. For example, Ramberg (1970) illustrated in detail an analysis in which displacements and stresses at each layer interface were matched to give the sinusoidal solution for the deflection of each interface, a more exact approach but one for which the outcome of layer parallel contraction is restricted to folding. The present paper builds upon Biot's mathematical analysis and reveals the deeper implications for instability and thus for structure development, during the deformation of a rigidly confined block of material.

It is widely recognized that Biot's analysis is applicable to materials which have an *intrinsic anisotropy* (i.e. those which have an anisotropy when the material is free of stress) such as well laminated or foliated rocks, and can explain the development of internal buckle folds in these materials. However, the analysis can also be applied to nonlinear isotropic materials and thereby provide an explanation for the production of localized shear bands or even discrete shear-failure surfaces in some granites and massive sandstones. The consequences of nonlinear material properties leading to *induced anisotropy* have been largely neglected in instability analyses. Thus kinking, a localized phenomenon, is inadequately explained using an internal instability analysis for linear materials.

The problems that previous workers have encountered when attempting interpretations of Biot's analysis can be traced to two geologically unrealistic assumptions built into the analysis. (1) The analysis assumes that the resistance to bending of the material is negligible. This will not be so for real geological materials made up of individual layers or fabric elements. (2) It is assumed that the material is linear. It is known however, that nonlinear material behaviour is important in the deformation of rocks and, as will be shown, can have a profound effect on the degree to which an incrementally anisotropic response is induced. The instability analysis presented in this paper takes into account the bending resistance of the material and the nonlinear material properties.

To begin the analysis, an example of constitutive relations applicable to Biot's theories for "anisotropic elastic" materials is described. One specific example, a power-law elastic model, is chosen to illustrate the most important mechanical effects of nonlinear properties, effects that are probably typical of many elastic–plastic strain-hardening rocks. Next, the effective modulus equations are derived for a simple bilaminate consisting of layers with properties described by the power-law elastic model. These equations define the mechanical properties of a homogeneous anisotropic continuum and are equivalent to the average properties of the nonlinear bilaminate multilayer. An expression for the average bending resistance of such a multilayer yields a bending coefficient which takes into account the number of layers in the confined multilayer. This coefficient,

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when added to the characteristic equation for internal instability, completes the groundwork for the new analysis. Solutions for the dominant characteristic direction for shearing are derived and presented graphically in terms of the degree of incremental anisotropy and the bending coefficient. The closing section is a general discussion around the central conclusion, that *pervasive internal buckling* and *oblique localized shearing* are the pure end-member types of internal instability. This conclusion differs from that reached by Cobbold *et al.* (1971) where kinking was considered to be a pure end-member of internal instability.

CONSTITUTIVE RELATIONS

Incompressible time-independent materials which are initially homogeneous and orthotropic (possibly isotropic) with respect to fixed reference co-ordinate directions x, y, z in some ground state are considered. The subsequent deformation is plane strain with in-plane loading parallel to the x and y directions. The material considered in this paper has a strain energy density, W, defined as a function of the maximum principal stretch $\lambda_1 = 1/\lambda_2 = \lambda$, or equivalently the maximum principal logarithmic strain, $\epsilon = \ln \lambda$. The material deforms homogeneously with principal stretches λ and $1/\lambda$ up to the point of instability or bifurcation. The finite stress/strain relation can be written

$$S_{xx} - S_{yy} = \frac{\lambda \mathrm{d}W}{\mathrm{d}\lambda} = \frac{\mathrm{d}W}{\mathrm{d}\epsilon}, \quad (1)$$

where S_{xx} and S_{yy} are the direct initial stresses acting in the x and y directions and $S_{xy} = 0$. Biot (1965, p. 101) showed that the incremental stress/strain equations would take the following form

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} \\ s_{22} - s &= 2Ne_{yy} \\ s &= 2Qe_{xy}, \end{aligned} \quad (2)$$

where $s = \frac{1}{2}(s_{11} + s_{22})$ and s_{11}, s_{22}, s_{12} $(=s_{21})$ are the incremental components of the Cauchy stress tensor referred to orthogonal axes (1, 2) rotated by the same amount as the local material has rotated; e_{xx} and e_{xy} are the infinitesimal strain components. The incremental stiffness moduli are

$$4N = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}(S_{xx} - S_{yy}) = \frac{\mathrm{d}^2 W}{\mathrm{d}\epsilon^2} \quad (3)$$

$$2Q = \frac{\lambda^4 + 1}{\lambda^4 - 1}(S_{xx} - S_{yy}) = (S_{xx} - S_{yy}) \coth(2\epsilon). \quad (4)$$

In (3), $4N$ is the tangent modulus for an in-plane uniaxial tension or compression test along x or y. Relation (4), derived by Biot (1965, p. 93) shows that the shear modulus Q can be expressed as a function of the initial stress $S_{xx} - S_{yy}$ and strain ϵ alone, whereas N cannot. N is a typical incremental coefficient in the sense that, for a stress/strain plot of a pure shear, the incremental modulus N, is given by the local slope of the curve at any prescribed stress or strain whereas Q has no obvious relationship with this curve.

This revelation, that the local slope of the stress/strain curve governs only one of the incremental moduli, will be seen to have important consequences for isotropic solids with finite stress/strain behaviour which is non-linear. This is because the incremental stress/strain behaviour governed by (2) i.e. the material response to non-uniform deformation, can have significant anisotropy induced ($N \neq Q$) during the uniform straining. This will now be illustrated using a nonlinear elastic isotropic constitutive model of the type investigated by Hutchinson & Tvergaard (1981). The model, though elastic, has often proved more successful at predicting critical conditions in buckling and necking problems in metals than equivalent models based on flow theories of plasticity.

Power-law elastic solid

The specific power-law model has the following strain energy function in plane strain

$$W = \frac{k\epsilon^{m+1}}{m+1}, \quad (5)$$

where m is a 'hardening' exponent which falls in the range $0 < m \leqslant 1$ and k is the linear stiffness coefficient. In plane strain tension $S_{xx} = P$, $S_{yy} = 0$ (or in compression where ϵ is taken as negative, $S_{xx} = -P$, $S_{yy} = 0$). From (1), (3), (4) and (5), the initial stress and incremental moduli become

$$P = k\epsilon^m \quad (6)$$

$$Q = \tfrac{1}{4}kq\epsilon^{m-1} \quad N = \tfrac{1}{4}km\epsilon^{m-1} \quad (7)$$

where

$$q(\epsilon) = 2\epsilon \coth 2\epsilon. \quad (8)$$

In order to facilitate a simplified analysis of multilayer behaviour under initial stresses, Biot's (1965) alternative formulation for incremental stresses will now be introduced. The alternative stress components t_{11}, t_{22}, t'_{12} $(\neq t'_{21})$ refer forces to initial areas and to locally rotated axes (1, 2). (The nonsymmetric Biot stress tensor $\mathbf{t}'$ to which these alternative stress components belong is obtained from the First Piola–Kirchhoff stress tensor $\mathbf{p}$ by forming the scalar product $\mathbf{p}\cdot\mathbf{R}$ where $\mathbf{R}$ is the orthogonal rotation tensor). The stress/strain equations for the components t_{11}, t_{22} and t'_{12} can be expressed as

$$\begin{aligned} t_{11} - t_{22} &= 4Me_{xx} \\ t'_{12} &= 2Le_{xy} \end{aligned} \quad (9)$$

with moduli

$$L = Q + P/2, \quad M = N + P/4. \quad (10)$$

The physical significance of these moduli has been discussed by Biot (1965, pp. 86 and 138). Using (6), (7), (8) and (10), L and M become

$$L = \frac{1}{4}k\epsilon^m\left(\frac{q}{\epsilon} + 2\right) \quad M = \frac{1}{4}k\epsilon^m\left(\frac{m}{\epsilon} + 1\right) \quad (11)$$

for the power-law elastic material.

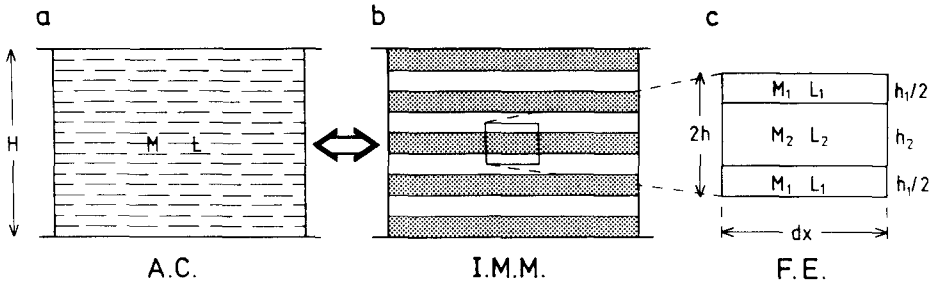

Fig. 1. The Idealised Multilayer Model (I.M.M) and its equivalent Anisotropic Continuum (A.C.) and representative Fundamental Element (F.E.). See text for details.

For $0.2 \leq m \leq 1$ and $\epsilon \leq 0.1$,

$$\frac{M}{L} \simeq m. \tag{12}$$

This approximation is valid to within a factor of 0.25 over the quoted range of m and ϵ (see Latham 1983, table 2.1).

From these mathematical developments [equations (5)–(12)] it may be concluded that an intrinsically isotropic material will have an incrementally anisotropic response when the stress/strain behaviour is nonlinear ($m \neq 1$ in the quoted power law). Biot (1965, p. 89) used the term "induced incremental orthotropy" (anisotropy) to describe this effect. As suggested by (12), the magnitude of the induced anisotropy is intimately related to the hardening exponent for a power-law material.

EFFECTIVE MODULUS EQUATIONS

These will be used to describe the average material properties of the Idealized Multilayer Model (I.M.M., Fig. 1) under initial stress.

Biot (1965, pp. 184–191), seeking a simplified stability analysis for multilayered media, considered how and to what extent a discretely laminated medium could be treated mechanically as though it were a homogeneous and anisotropic continuum. Biot's treatment of the problem uses what is known as an effective modulus theory (see Ting 1980, for a review of theories of composites). It is similar to that given by Postma (1955) although Biot's equations considered the laminated medium to be under initial stress and are therefore more useful for analysing static buckling behaviour in contrast to dynamic stability problems. The effective moduli that describe the average behaviour of a periodic bilaminate, given by equations (14) and (15) below, provide the essential link between many of Biot's theories of buckling and their applications to structural geology, particularly problems of folding (e.g. Biot 1964, 1965a,b,c, 1967, Johnson 1970, 1977, Cobbold *et al.* 1971, Summers 1979, Williams 1980).

The effective modulus equations which describe the average constitutive response of the smallest typical element (Fig. 1c) of the stressed I.M.M. (Fig. 1b) are (Biot 1965, 1967)

$$P = P_1\alpha_1 + P_2\alpha_2 \tag{13}$$

$$M = M_1\alpha_1 + M_2\alpha_2 \tag{14}$$

$$L = \frac{L_1 L_2}{\alpha_1 L_2 + \alpha_2 L_1} \tag{15}$$

$$\alpha_1 = \frac{h_1}{h_1 + h_2}, \quad \alpha_2 = \frac{h_2}{h_1 + h_2} \tag{16}$$

given the following definitions and restrictions.

(1) The alternating layers labelled 1 and 2 are homogeneous, incompressible, elastic and have interfaces which are coherent and parallel to any plane of orthotropic symmetry that may exist within each of the materials should they not be isotropic.

(2) Layers of thickness h_1 and incremental elastic moduli M_1, L_1 alternate with layers of thicknesses h_2 and moduli M_2, L_2.

(3) The initial stresses are oriented with principal directions along x and y (i.e. $S_{xy} = 0$). The effective compressive initial stresses ($S_{yy} - S_{xx}$) within layers 1 and 2 are P_1 and P_2, respectively.

(4) The parameters α_1 and α_2 are termed the fractional thicknesses as they represent the fraction of the total thickness occupied by each material, ($\alpha_1 + \alpha_2 = 1$).

(5) The characteristic wavelength of any perturbations in the displacement field that develops in the composite material is much larger than the lamination thickness so that the state of stress and strain may be assumed constant within each layer of the fundamental element.

Biot's derivation of (14) and (15) is via the incremental stress components t_{11}, t_{22} and t'_{12} which are by definition linearly related to the incremental strain components e_{xx}, e_{yy} and e_{xy} for incrementally linear time-independent materials. In obtaining (15) he used the important result that the same incremental shear stress t'_{12} acts upon each layer interface of the fundamental element and throughout the composite (Fig. 3), whereas the component s_{12} is not the same within each layer. Replacing the L's by Q's in (15) would therefore give an incorrect average shear modulus Q, whilst replacing the M's by N's would still give a valid equation for the average direct modulus, N, as both t_{22} and s_{22} are the same within each layer. Note that incompressibility ensures that $\alpha_1 = h'_1/(h'_1 + h'_2)$ in Fig. 3.

The unusual decomposition (see Biot 1973, appendix) of the incremental deformation shown in an exaggerated form in Fig. 2 is valid for infinitesimal strains e_{xx}, e_{yy}, e_{xy} and an infinitesimal rigid body rotation. The deformation states shown in Fig. 2 correspond to (a) uniform (finite) shortening just prior to the onset of non-uniform

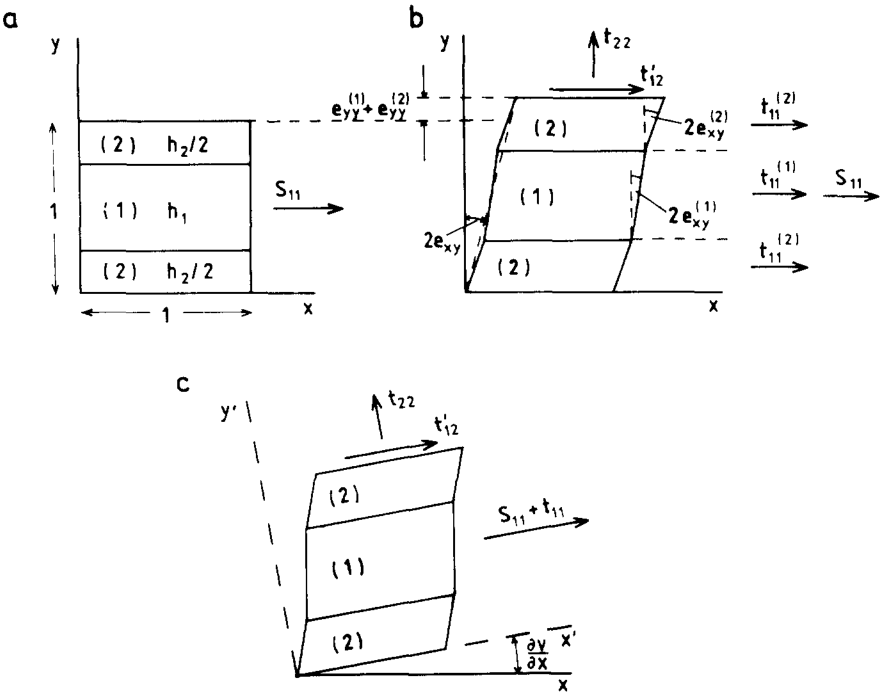

Fig. 2. Deformation states assumed for the general deformation of the Fundamental Element. See text for discussion.

deformation; (b) that part of the incremental deformation which is due to physical internal straining of the material and assumed to be equivalent to a superposed pure shear and simple shear (in either order) both parallel to the layer boundary, and (c) the total deformation, as in (b) but with the added geometric effect of the local rigid-body rotation $\partial v/\partial x$ of both the material and the entire stress field acting upon it.

From substitution of the power-law elastic model for the material properties of layers 1 and 2 of the I.M.M., the effective moduli describing the average anisotropic response of the fundamental element may be determined from (10), (11), (13), (14) and (15) as

$$L = \frac{1}{4} k_2 \left\{ \frac{k_r \epsilon^{m_1+m_2}(q/\epsilon + 2)}{\epsilon^{m_2}\alpha_1 + k_r \epsilon^{m_1}\alpha_2} \right\} \qquad (17)$$

$$M = \frac{1}{4} k_2 \left\{ k_r \epsilon^{m_1}\left(\frac{m_1}{\epsilon} + 1\right)\alpha_1 + \epsilon^{m_2}\left(\frac{m_2}{\epsilon} + 1\right)\alpha_2 \right\}. \qquad (18)$$

The differential initial stress P is given by

$$P = k_2(k_r \epsilon^{m_1}\alpha_1 + \epsilon^{m_1}\alpha_2) \qquad (19)$$

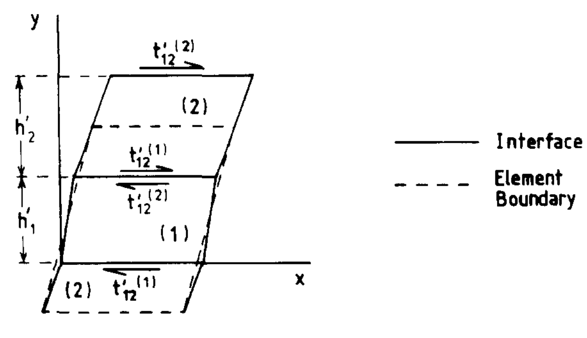

Fig. 3. Interface incremental shear stresses acting on the layers of the Fundamental Element.

and the subscripts refer the parameters to layers 1 and 2. In the stability analysis which follows, it is the ratio $k_r = k_1/k_2$ rather than the absolute values of k which is significant.

It is important to emphasize the scope afforded by these effective modulus equations. The significance of the ratio M/L, that is, the incremental anisotropy of an anisotropic continuum [given by (17) and (18)], lies in its application to many of Biot's theories of instability in multilayers. It is now clear that the incremental anisotropy M/L in multilayers with nonlinear material properties considered above includes both the influence of intrinsic anisotropy (that may be loosely associated with 'competence contrast') and the influence of the induced anisotropy from within each layer. An initially isotropic relationship between incremental stresses and strains within individual layers becomes increasingly anisotropic while the deviatoric state of the initial stress builds up as a direct result of the nonlinearity of the material in that layer. The initial stress is said to *induce* a material response which is incrementally anisotropic, a physical effect which is discussed briefly at the end of this paper. Biot's analyses of stability in anisotropic media (Biot 1965, Chapter 4) which are designed to cope with incrementally anisotropic material responses are independent of the source of the incremental anisotropy. As pointed out by Sowers (1973), it may be intrinsic, induced, or a combination of both.

It should be noted that equations (17), (18) and (19) are only applicable to elastic–plastic deformations up to the point of incipient instability. The parallelism of the orthotropic symmetry in the total stress field and in the material properties breaks down soon after this point of instability develops.

Some valuable insights can be gained by applying the

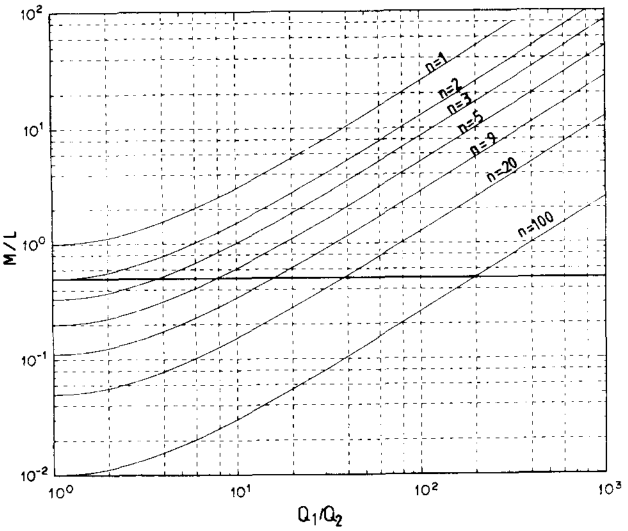

Fig. 4. Approximate representation of Incremental Anisotropy, M/L, as a function of shear modulus ratio (competence contrast), Q_1/Q_2, and average nonlinearity, n, of the material properties in each layer, for multilayers with a constant thickness ratio $\alpha_1 = 0.5$ and identical stress exponents in each layer (i.e. $1/m_1 = 1/m_2 = n$). The line $M/L = 0.5$ ($=10^{-0.3}$) separates the fields for first-kind and second-kind instability.

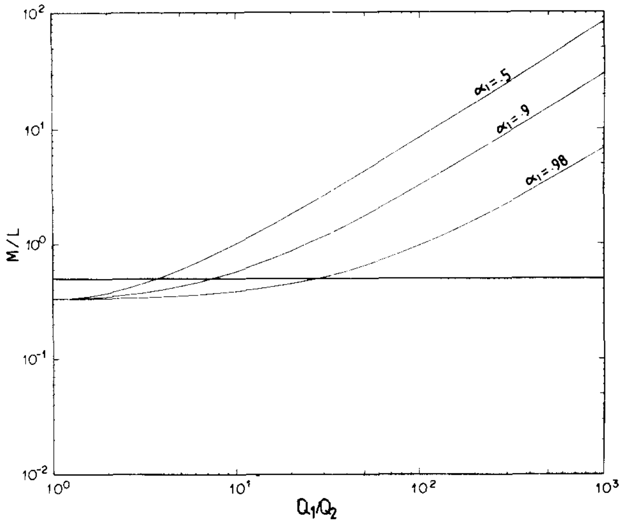

Fig. 5. Approximate representation of Incremental Anisotropy, M/L, as a function of shear modulus ratio (competence contrast), Q_1/Q_2, for different thickness ratios and a constant stress exponent n ($=1/m_1 = 1/m_2$) of 3.

approximation (12) for the properties of each layer. Further, if each layer has an identical stress exponent n ($=1/m_1 = 1/m_2$), then there are two useful plots (Figs. 4 and 5) for M/L vs shear modulus ratio or 'competence contrast' Q_1/Q_2 ($=k_r$ for these conditions).

In Fig. 4, the layers have equal thicknesses ($\alpha_1 = 0.5$) and the plot indicates that for increasingly nonlinear material properties (higher n values), the incremental anisotropy takes progressively lower values. Just as incremental anisotropy is induced when $n > 1$ for a homogeneous isotropic material, i.e. when $k_r = 1$, so too is incremental anisotropy induced in the average properties of a multilayer ($k_r \neq 1$) for $n > 1$. Nevertheless, a large ratio k_r which is generally associated with intrinsic anisotropy will usually overshadow the effect of induced anisotropy giving an average incremental anisotropy of the multilayer, M/L, which is greater than 0.5.

If α_1 is increased so that the layers have unequal thicknesses, M/L is lowered. This is demonstrated for a value of $n = 3$ in Fig. 5. In interpreting Figs. 4 and 5, it should be recognized that the full complexity of the strain dependence of equations (17) and (18) is not represented but will be accounted for in a numerical analysis to follow in a companion paper.

RESISTANCE TO BENDING

For anisotropic media comprising fine-scale alternations of layers of differing incremental moduli, Biot (1967) calculated an approximation for the resistance to bending representative of the fundamental element, due to its fine layered structure

When a material exhibits incremental anisotropy (i.e. $M \neq L$), the effect of a resistance to bending may be represented by a couple $\mathcal{M}$ per unit area acting on a surface initially perpendicular to the reference x-direction (see Fig. 6). The value of this stress couple is assumed to be proportional to the curvature acquired by fibres of the medium initially parallel to the x-direction. Hence Biot expressed

$$\mathcal{M} = \frac{b\,\partial^2 v}{\partial x^2}, \tag{20}$$

where, for a displacement v in the y-direction, $\partial^2 v/\partial x^2$ is a good approximation to the curvature for slopes of less than about 10 or 20° (see Johnson 1970, for details of this approximation to curvature). Biot called b the *couple stress coefficient*. It is a measure of the 'bending rigidity' of the medium. (In the example of a finite element of a thin elastic bending beam, it is related to the product of Young's modulus, E and the moment of inertia, I of the element).

Biot (1967) stated that the presence of couple stresses implies non-symmetric shear stress components $t'_{12} \neq t'_{21}$. However, the difference between these two stress components exists even when there is no internal resistance to bending. This non-symmetry follows from the condition that the total torque acting on the deformed element must be zero (Biot 1965, p. 61) whereas the stress components t'_{12}, t'_{21} represents forces acting on an element of unit dimensions before deformation. The existence of a couple stresses imparts an additional degree of asymmetry as illustrated by the two terms on the R.H.S. of the second of the equilibrium equations (22) below.

Equilibrium of an element of the medium about an axis perpendicular to the page in Fig. 6 implies the moment equation

$$t'_{12} - t'_{21} = \frac{\delta\mathcal{M}}{\partial x}. \tag{21}$$

Equilibrium in the vertical direction (i.e. perpendicular to x) yields the following equations for the stress components of t'

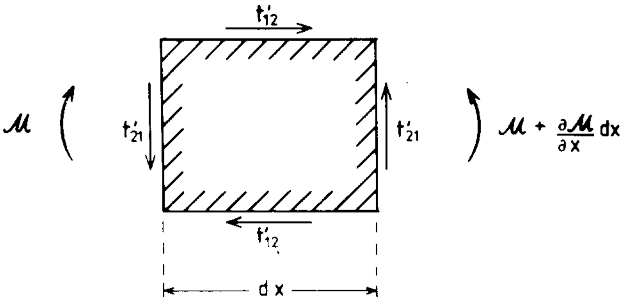

Fig. 6. Physical interpretation of equilibrium of moments, see equation (21) (after Biot 1967).

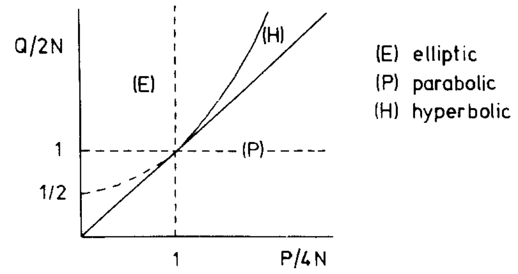

Fig. 7. Instability (or bifurcation) regimes of equation (28) as classified by Hill & Hutchinson (1975).

$$\frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} = 0$$

$$\frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} = \frac{P\,\partial^2 v}{\partial x^2} + \frac{\partial^2 \mathcal{M}}{\partial x^2}. \quad (22)$$

The bending resistance introduces a term of fourth order in $\partial v/\partial x$ into these equilibrium equations. (The equivalent term may be introduced into the variational principle expression for the energy density, see Biot 1967, 1974).

Assuming a sinusoidal solution for the displacement v in equation (20), Biot (1967) derived the following expression for the couple stress coefficient b of the fundamental element in Fig. 1(c)

$$b = \frac{4}{3} h^2(\alpha_1^2 M_1 - \alpha_2^2 M_2) \frac{(L_1 - L_2)\alpha_1\alpha_2}{\alpha_1 L_2 + \alpha_2 L_1}. \quad (23)$$

If layer (1) is much more 'competent' than layer (2), i.e. if $M_1 \gg M_2$, $L_1 \gg L_2$ equation (23) reduces to the approximation

$$b = \tfrac{1}{3} h_1^2 M_1. \quad (24)$$

TYPES OF SOLUTIONS FOR THE DISPLACEMENT FIELD—DISCUSSION

It has been shown from the theory developed so far, that the stress/strain equations (9) can be applied, as an approximation, to multilayers with effective moduli such as (17) and (18) as well as to homogeneous possibly nonlinear intrinsically isotropic materials. When combined with equilibrium equations, the resulting displacement field equations [e.g. equation (26)] have broader applications than those for which the two material moduli (e.g. M and L) refer to the behaviour of single component materials. Notice, for example, that by setting material constants of alternate layers to be equal, the exact single component material properties are also given by the effective moduli.

Neglecting resistance to bending within the fundamental element (Fig. 1), the differential equation for the function ϕ describing the non-uniform displacements, where $u = \partial\phi/\partial y$, $v = -\partial\phi/\partial x$ is (e.g. Biot 1965, Hill & Hutchinson 1975, Johnson 1980)

$$(Q - P/2)\frac{\partial^4\phi}{\partial x^4} + 2(2N - Q)\frac{\partial^4\phi}{\partial x^2\partial y^2} + (Q + P/2)\frac{\partial^4\phi}{\partial y^4} = 0 \quad (25)$$

or in terms of the moduli M and L, this becomes

$$(L - P)\frac{\partial^4\phi}{\partial x^4} + 2(2M - L)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{L\partial^4\phi}{\partial y^4} = 0. \quad (26)$$

Johnson (1980) considered a sinusoidal form for the general solution of the function ϕ as he was concerned with the buckling of single layers with elastic–plastic strain hardening material properties. (See also Biot 1965, p. 218.)

An alternative form may be chosen for the general solution of the function ϕ. It is written

$$\phi = f_1(x + \xi_1 y) + f_2(x - \xi_1 y) + f_3(x + \xi_2 y) + f_4(x - \xi_2 y). \quad (27)$$

Each function f_1 to f_4 is an analytic or non-analytic function of a heterogeneous simple shear parallel to planes along which $x \pm \xi y$ = constant, so that ξ is associated with a direction making an angle θ with y, where $\theta = \tan^{-1}\xi$. These are the characteristics or characteristic lines discussed by Johnson & Ellen (1974) and by several other workers. This choice for the function ϕ leads to a simple classification of the solutions of (25) into three regimes depending upon the current values of N, Q and P. Substituting (27) into (25), the characteristic equation describing the onset of instability due to real and/or imaginary characteristics (i.e. heterogeneous simple shear displacement patterns) becomes

$$(Q - \tfrac{1}{2}P)\xi^4 + 2(2N - Q)\xi^2 + (Q + \tfrac{1}{2}P) = 0. \quad (28)$$

This equation (28) was classified by Hill & Hutchinson (1975) into elliptic, parabolic and hyperbolic regimes as shown in Fig. 7.

In Fig. 7 (cf. Cobbold *et al.* 1971, fig. 2), the instability regimes (E), (P) and (H) each correspond to different types of non-uniform displacement fields that can develop in a material. The type which develops will depend on the current values of the stress difference P and the moduli N and Q. Subject to the restrictions $N > 0, Q > 0, P > 0$ these regimes are (Hill & Hutchinson 1975)

(E): no real ξ, $2N > Q - \sqrt{(Q^2 - P^2/4)}$
(P): 2 real ξ, $Q < P/2$

(H): 4 real ξ, $2N < Q - \sqrt{(Q^2 - P^2/4)}$.

The (E/P) interface corresponds to Biot's condition for a vanishing small wavelength internal-buckling mode, a special degenerate case of what he calls *First Kind Instability* (Type 1 instability of Cobbold *et al.* 1971). This occurs when the compressive differential stress P first meets the value of the slide modulus L (Biot's 'Shear Threshold'). The (E/H) interface corresponds to Biot's condition for *Second Kind Instability* (Type 2 instability of Cobbold *et al.* 1971) and occurs when P first meets $4(M/L)(L - M)$. On (E/H), the four real roots represent heterogeneous simple shearing along characteristic directions $\pm\theta(=\tan^{-1}\xi)$, and they coincide in pairs. (Within (H) all the directions may be different.)

Although Biot's analysis (1965) has been known to geologists for almost 20 years, certain fundamental points remain poorly understood. From a theoretical consideration of the anisotropic continuum analysis leading to (28), Biot's first kind and second kind instabilities are in fact equally 'localized' in the sense that they both involve real characteristics parallel to which there is heterogeneous simple shear. However, because an analysis producing instability in the (P) regime invariably applies to those statistically homogeneous composite materials in which the bending of planar elements occurs, the sinusoidal waveform (i.e. the lowest harmonic) for the heterogeneous simple shear is the most apposite choice at low amplitudes. The propagation of this 'localized' waveform parallel to either or both characteristics, then leads to Biot's "Internal Buckling". This is because the boundary conditions cause reflections and an interference pattern results. It appears instantaneous giving an equilibrium configuration only because the analysis is totally elastic. For real elastic–plastic materials, where the average bending resistance due to planar elements is significant, rapid propagation can be expected and instability in the (P) regime for confined materials will tend to produce pervasive and repetitive rather than localized buckling disturbances by a process of reflection. Consider now, the possibilities for less restrictive boundary conditions.

The non-uniform modes of displacement associated with (E) (Fig. 7) are generally taken to be, although they need not necessarily be, sinusoidal. If they are sinusoidal, they have an exponential decay in one direction. These modes are often referred to as diffuse modes as they do not permit localized simple shearing in bands. For example, in an unconfined layer or block of material, the change in equilibrium from a uniform shortening under compression to a non-uniform buckling or barrelling mode of displacement is a problem of instability involving a bifurcation into the (E) regime and it is solved not by seeking real characteristics for (25), but by seeking antisymmetric and symmetric sinusoidal solutions with exponential decay (see Biot 1965, pp. 204–213, Young 1976, Johnson 1980 and for similar analyses using nonlinear viscous fluids see Fletcher 1974, Smith 1977).

Johnson's (1980) *analysis of folding and faulting*

The conditions of instability that were investigated by Johnson (1980) (Fig. 7) correspond with layer-parallel compression of a single layer (of rock) within a non-rigid confining medium (of other rocks). Depending upon the boundary conditions at the layer interfaces and the material properties of the layer and confining medium, he argued that the layer folded when solutions of (28) were in the (E) or (P) regime. However, when boundary conditions strongly resisted the deflection of the layer interfaces, solutions of (28) moved into the (H) regime before fold amplification in the (E) and (P) regimes was significant. Because all the characteristics are real in (H), there can be various discontinuities and Johnson suggested this was the condition for faulting. He also extended his analysis to multilayers with perfect or near perfect slip between layers using a thin-plate approach to the the stability problem and considered the effect of having different numbers of layers in the multilayer stack. (This will be discussed in a companion paper.)

In this paper, initiation of structures in multilayers is also analysed by examining regimes of the characteristic equation (28), using expressions (17), (18) and (19) for L, M and P. To restrict the problem to manageable proportions, rigid rectangular frictionless boundary conditions are assumed with the result that stability prevails in the (E) regime. As $P/4N$ increases, exit from the (E) regime (Fig. 7), resulting in an internal instability of the first or second kind, is either at the (E/H) interface or the (E/P) interface depending upon whether $Q/2N$ is greater or less than 1. The simple classification of equation (28) into regimes (E), (P) and (H) would appear to be a powerful tool for many plane-strain problems.

INTERNAL INSTABILITY IN MEDIA WITH BENDING RESISTANCE

Introducing the 'bending correction' in the equilibrium equations (22), and using (9), the new characteristic equation [cf. equation (28)] becomes (Biot 1967)

$$P = L\left\{1 + \frac{a}{\xi^2} + 2\,\frac{(2M - L)}{L}\,\xi^2 + \xi^4\right\} \qquad (29)$$

where

$$a = \frac{\pi^2 b}{LH^2} \qquad (30)$$

and L, M and P for the case of the power law I.M.M. are given by (17), (18) and (19).

The parameter a is the confined medium bending coefficient and H is the confinement distance (Fig. 1). In defining a it is also assumed that the parameter ξ is given the geometric interpretation associated with the wavelength ratio of internal buckling i.e. $\xi = \mathcal{L}_x/\mathcal{L}_y$ and that the wavelength in the y-direction, $\mathcal{L}_y$, equals $2H$. Further, it is assumed that the layering extends infinitely in the x-direction and solutions that permit the condition

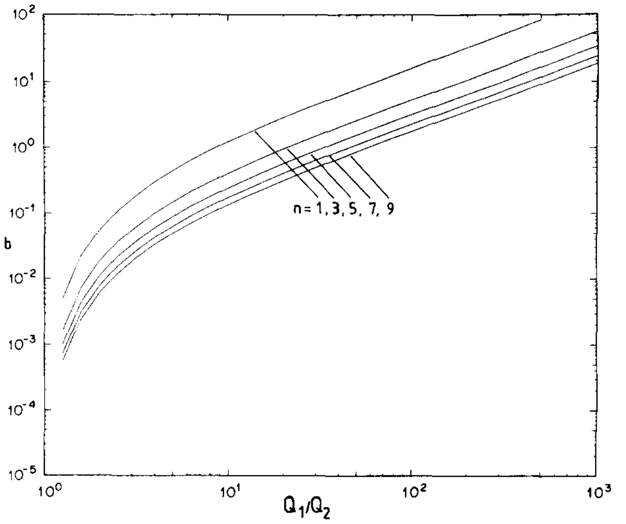

Fig. 8. The bending rigidity b of the fundamental element as a function of the shear modulus ratio (competence contrast), Q_1/Q_2 and stress exponent n ($=1/m_1 = 1/m_2$) for a constant thickness ratio $\alpha_1 = 0.5$ and unit layer thickness ($h = 1$) assuming the approximation (12) for the properties of each layer.

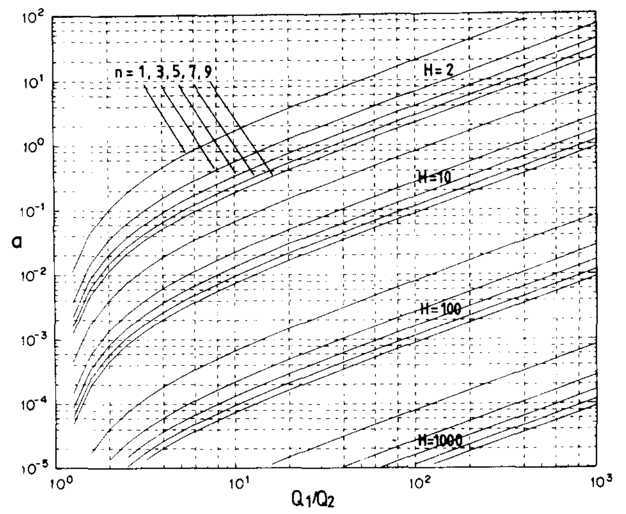

Fig. 10. The confined medium bending coefficient a as a function of shear modulus ratio (competence contrast) Q_1/Q_2, stress exponent n ($=1/m_1 = 1/m_2$) and the total number of competent and incompetent layers in the confined multilayer H, for a constant thickness ratio $\alpha_1 = 0.5$, assuming the approximation (12) for the properties of each layer.

which Biot (1964) refers to as self-confinement are disregarded for the present investigation.

The term a/ξ^2 in (29) is due to the couple stress and plays a crucial role in that the maximum value of P does not coincide with $\xi = 0$ for first-kind instability. This term also contains the geometric parameters H and h. If h, the average layer thickness, is set equal to unity, then the confinement distance H is also equal to the number of layers. This normalizing procedure, if interpreted carefully, allows a comparison of the essential features of multilayers on all scales up to those where gravity becomes a significant factor.

Consider the bending rigidity of a fundamental element with equal-thickness layers ($\alpha_1 = 0.5$) and equal stress exponents in the power law for each layer ($1/m_1 = 1/m_2 = n$). Figure 8, which incorporates approximation (12) shows that higher values of the contrast in shear moduli Q_1/Q_2 ($=k_1/k_2$) and lower values of n tend to increase the bending rigidity.

For internal instability, it is also necessary to know whether the fundamental element occupies a large or small part of the total confined multilayer; that is, whether there are few layers (small H) or many layers (large H). If there are many rather than few, we may expect a given volume V of the confined multilayer to be less resistant to buckle initiation because the boundaries of the multilayer which inhibit shear mode buckling are less in evidence. The bending rigidity b of the medium is identical in Figs. 9(a) and (b) since the fundamental elements are identical. However, permissible buckling wavelengths are not identical for the same level of initial stress. It is therefore a rather than b that is incorporated into the buckling condition, a condition involving resistance to layer parallel shear as well as resistance to bending of individual components.

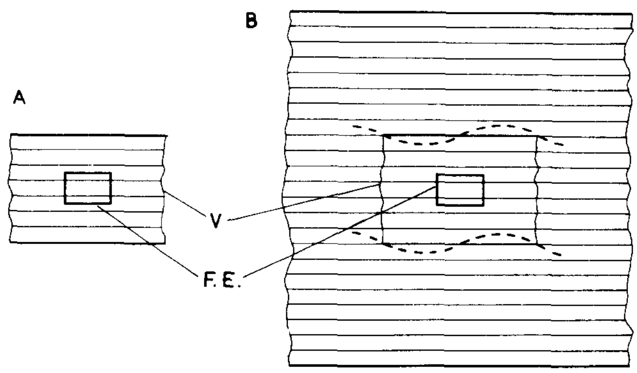

Fig. 9. Illustration of the bending constraints imposed by assuming different numbers of layers in the confined multilayer. The bending rigidity b is identical in A and B. The confined medium bending coefficient, a is higher in A than in B. See text for discussion.

In Fig. 10 the confined medium bending coefficient a has been plotted against Q_1/Q_2 for selected values of H. A family of curves with the same selection of n ($=1/m_1 = 1/m_2$) values is given for each number of layers (H) considered, with α_1 set equal to 0.5. Using logarithmic axes, the families are identical but shifted bodily to lower values of a for increasing values of H. For $H = 2$, the confined multilayer is equivalent to the fundamental element only. The bending resistance will have most influence on the form of the instability when the term a has a greater value. This occurs when there are fewer layers in the confined multilayer. Note that a will rarely exceed a value of 0.1.

RESULTS—DOMINANT CHARACTERISTICS

Equation (29) is the characteristic equation of internal instability for an anisotropic continuum of time-independent material properties with internal resistance to bending. The conjugate characteristic directions of heterogeneous simple shear that offer the least resistance to 'layer' parallel compression and are therefore the most likely to develop can be determined from this equation.

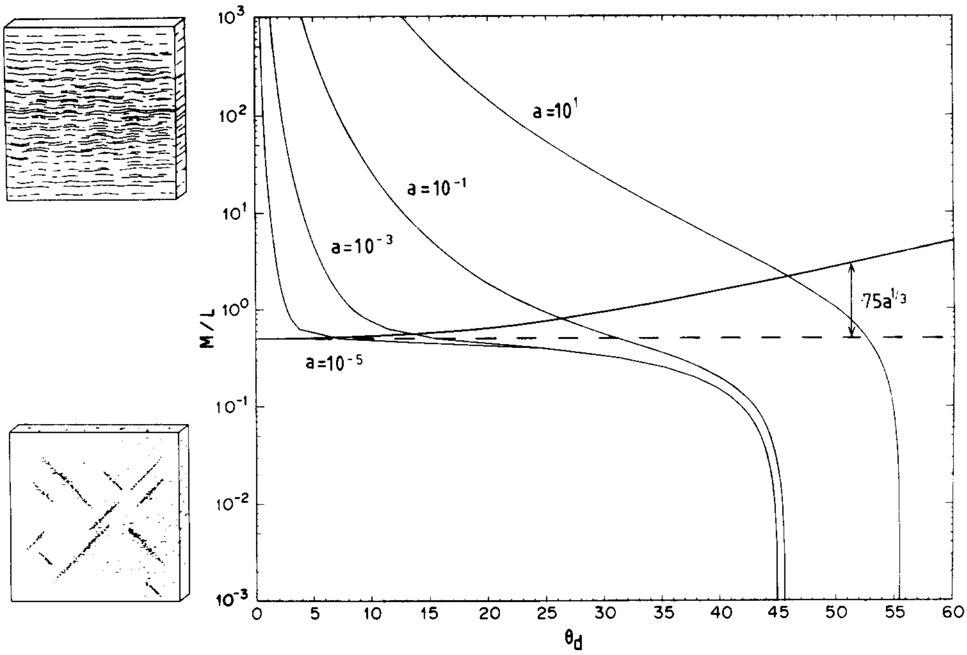

Fig. 11. The relationship between the incremental anisotropy M/L, the confined medium bending coefficient a and the dominant characteristic direction θ_d at the point of incipient internal instability.

Method

Differentiation of the bracketed term of the R.H.S. of equation (29) with respect to ξ and setting the result equal to zero yields a sixth order equation in ξ_d which reduces to a cubic in ξ_d^2. The positive real root ξ_d^2 of this equation can be found using a combination of a trigonometric method and Cardan's method for solving cubic equations. It represents the value of ξ which corresponds to a minimum value of P and is therefore the dominant characteristic ξ_d. Due to the physical restrictions on the signs of a and g (equation (35) below), there exists either one or three real roots of ξ_d^2 in each case. The solutions that are obtained for ξ_d are given below

$$\xi_d^2 = 1/f_1 \quad q < 0 \tag{31}$$

(three real roots of ξ_d^2 exist only one of which is positive),

$$\xi_d^2 = 1/f_2 \quad q > 0 \tag{32}$$

(one real root of ξ_d^2 exists which is always positive, two complex), where

$$q = \left(\frac{r}{3}\right)^3 + \left(\frac{c}{2}\right)^2 \tag{33}$$

$$r = -\frac{2g}{a} \tag{34}$$

$$g = \frac{2M - L}{L} \tag{35}$$

and a was given in (30),

$$c = -\frac{2}{a} \tag{36}$$

$$f_1 = 2\sqrt{-r/3}\,\cos(\gamma/3) \;\text{ where }\; \cos\gamma = \frac{-c}{2\sqrt{-(r/3)^3}} \tag{37}$$

$$f_2 = A + B \tag{38}$$

$$A = \sqrt[3]{\frac{-c}{2} + \sqrt{q}} \tag{39}$$

$$B = \sqrt[3]{\frac{-c}{2} - \sqrt{q}}. \tag{40}$$

It was suggested earlier that any value of ξ may be described in terms of a direction $\theta = \tan^{-1}\xi$ in degrees and therefore a more useful graphical parameter at least for geological purposes is the dominant characteristic direction θ_d. These directions occur in conjugate sets given by $\pm\theta_d$ for the high symmetry condition of this 'layer-parallel compression' analysis.

The theoretical relationships presented in Fig. 11 describe the characteristic directions at the first appearance of instability for an increasing applied stress P. By examining the conditions for which positive real roots of ξ_d^2 exist, it is suggested that there are two different instability fields of relevance to the present problem and that these have the same significance as the two fields associated with internal instability in materials assumed to have no bending resistance. However, in the analysis which takes into account bending resistance, the instability fields are not separated by the conditions $M/L = 0.5$. The condition separating the two fields can be determined by substituting $q = 0$ into equation (33) and using (34) and (35). This gives

$$g^3 = \frac{27a}{8}. \tag{41}$$

Substituting equation (35) into (41) yields

$$\frac{M}{L} = \frac{1}{2} + \frac{3a^{1/3}}{4} \tag{42}$$

which is the line separating the two instability fields. Its approximate position is indicated in Fig. 11.

It will be assumed, then, that when the applied stress reaches the critical level for internal instability, the material conditions for first- and second-kind instability are respectively,

$$\frac{M}{L} > 0.5 + 0.75a^{1/3}$$

$$\frac{M}{L} < 0.5 + 0.75a^{1/3}.$$

A useful parameter, R which will be referred to in a companion paper can now be defined

$$R = (M/L)/(0.5 + 0.75a^{1/3}). \quad (43)$$

It may be interpreted as an expression of the degree to which the roots of the characteristic equation are associated with first- or second-kind instability. For $R > 1$, of the three real roots of ξ_d^2, only one is positive. For $R < 1$, there is only one real root of ξ_d^2 which is always positive; the other two are complex.

DISCUSSION AND CONCLUSIONS

The most important general results of taking into account bending resistance in the internal instability analysis are summarized in Fig. 11. The complexity of the strain-dependent parameters M/L, a and P for the power-law elastic idealized multilayer model will be temporarily disregarded to simplify the discussion. It should however be noted that M/L and a in the figure refer to values at the critical strain for incipient internal instability which is by differential shearing along the dominant characteristic directions, θ_d.

For any real composite material analysed using a homogeneous anisotropic continuum approach, the average bending rigidity could be an essential 'microstructural' correction for a description of the mechanical stability of that continuum. If the effect of bending resistance is neglected, setting $a = 0$, there are two distinct branches to the curve of M/L vs θ_d. For $M/L > 0.5$, $\theta_d = 0$ and there is first-kind instability. For $M/L < 0.5$, θ_d varies from 0 to 45° and there is second kind instability. These branches were shown in the curve given by Cosgrove (1976, fig. 5).

For the realistic case of non-zero values of a, as a increases, the two originally distinct branches to the curve have a progressively smoother transition between the fields of first- and second-kind instability (Fig. 11) which is an important step towards understanding the hybrid form of many structures initiated in multilayers. For example, this may help to explain a common observation of multilayer experiments for which internal resistance to bending is relatively high: that the early development of fold limbs from internal buckles readily take on the appearance of localized oblique zones of shearing and these appear to propagate and reflect off the boundaries of the multilayers. Examples of such experiments are shown in Cobbold *et al.* (1971) and Johnson (1977). To argue whether these concentric-like fold structures initiate as buckles or as localized kinks would be missing the point. The introduction of a into the analysis illustrates that the non-uniform deformation associated with a particular θ_d is a result of relative contributions of first and second kind end-member behaviour. θ_d is the preferred orientation for the heterogeneous simple shear deformation and the contributions from the end members indicates to what extent a pervasive and harmonic internal buckling pattern (i.e. of superimposed reflecting and interfering sinusoidal functions of heterogeneous simple shear) is preferred to a more localized one. The figure helps to explain why localized shearing is more common at angles of 30–45° to the bulk extension direction than sub-parallel to it and conversely, why pervasive harmonic patterns of heterogeneous simple shear are most commonly developed sub-parallel to the bulk extension direction.

The value of M/L for which the internal buckling and localized shearing end members are in equal proportion is given by the intermediate line $M/L = 0.5 + 0.75a^{1/3}$ (Fig. 11). This suggests then when a is large as a result of there being very few confined layers (see Fig. 10), shear localization may be expected even when M/L is greater than 0.5 at instability.

Some indication of the role played by multilayer parameters (e.g. material properties, proportions and numbers of each layer) upon the value of M/L was given in Figs 4, 5 and 10, from which the following conclusions can be drawn, using the result of Fig. 11.

When M/L at instability is much greater than the intermediate value (in most practical cases, 0.5), the initial competence contrast which is an intrinsic average property of the unstressed multilayer will have played the most important part in contributing to the degree of incremental anisotropy (M/L) at instability. This form of average incremental anisotropy (mostly intrinsic) is associated with planes that offer least resistance to shearing lying parallel to the initial compressive stress.

The significance of nonlinear material properties becomes apparent when M/L at instability is much lower than the intermediate value. In such cases, the individual layer(s) will have had incremental anisotropy induced during the initial uniform compression. This leads to a form of average incremental anisotropy which is associated with planes that offer least resistance to shearing lying at 45° to the initial compression at the point of instability.

A detailed treatment which considers the strain dependence of the multilayer parameters and the initial stress is the subject of a companion paper in which geologically realistic numerical examples are considered.

The analysis of internal instability presented in this paper may be adapted along the lines indicated by Cobbold *et al.* (1971) and Cosgrove (1976) to structures initiated during layer-parallel extension.

The mechanical explanations for the development of the finite banded perturbations classified as S-bands by Cobbold (1977a,b) and for the initiation of internal structures associated with heterogeneous simple shear (real characteristics) discussed in this paper are clearly related. Consideration of those structures that Cobbold classified as P or PS bands are excluded from the internal instability analysis by the incompressibility (or constant volume) condition which forbids a differential simple extension parallel to the band.

Induced incremental anisotropy

The neglected concept of induced incremental anisotropy applicable to the behaviour of isotropic materials deserves further consideration. Suppose a real block of initially isotropic material which exhibits nonlinear stress/strain behaviour has stresses applied to it in accordance with the conditions of the internal instability analysis. The physical consequences of the material nonlinearity that result from accumulating inelastic flow during the uniform pure shear are described mathematically by a change in the ratio of the moduli M and L. In turn, the change in ratio may be described as a decrease in the probability that coaxial deformation will continue within every element of the block for a further increment of applied shortening.

The type of physical phenomena that could account for this change upon loading is a subject beyond the scope of this paper. However, a suggestion that it is related to the physical processes that lead to the development of a corner at the loading point on the yield surface seems reasonable in the light of the current popularity of yield surface vertex models in applications of plasticity theory. (In fact, the behaviour described by the nonlinear elastic model chosen in this paper coincides with that represented by the total loading behaviour of the vertex solid model as described by Christofferson & Hutchinson 1979). Vertex solid models have been applied to the deformation of metals (see for example Hutchinson & Tvergaard 1981) and to the deformation of rock by frictional sliding on fissure surfaces (see Rudnicki & Rice 1975). Physical processes giving rise to 'non-normality' of the loading path at the yield surface (Needleman 1979) are thought likely to be related to pressure sensitive deformation mechanisms such as frictional sliding, and dilatancy. It appears then, that during the uniform deformation by continued stressing in the same direction, a selective unblocking of previously inactive slip systems in the polycrystal or fissured rock makes further coaxial deformation at all places within the block of material progressively less feasible. For example, it is not difficult to imagine that anisotropy would be induced in pressure sensitive dilatant materials with randomly oriented microcracks on which shearing takes place since the cracks would be subject to selective opening and closing during a bulk uniform stressing. Eventually, with continued stressing, the local effects of non-coaxial increments link up and lead to localized shearing within an inclined band or on a discrete surface. This is the reason why shear bands and shear fractures are common within isotropic rocks such as some granites and massive sandstones and folding is absent—there is nothing there to fold. Even multilayered rocks may deform unstably by oblique localized shearing (M/L at instability favours second-kind end member behaviour) if the layering plays a minor mechanical role.

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